A unified solution in fuzzy capital budgeting

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ABSTRACT

Since the mid-1980s, both academics and practitioners have proposed and discussed various solutions in fuzzy capital budgeting. Based primarily on traditional capital budgeting methods, these solutions present the same problems as their respective deterministic methods: the implicit assumptions of the reinvestment rates; the possibility of multiple rates of return; and the possibility of anomalous behavior of the net present value method. This paper presents a unified solution in fuzzy capital budgeting based on modified deterministic methods proposed in the financial literature. This unified approach eliminates these problems and has the property of matching decisions on acceptance or rejection of investment projects with same life horizons and same scales and therefore maximize shareholder wealth. An insight is provided into the advantages of these investment project appraisal methods by comparing and contrasting them with traditional fuzzy methods. A comprehensive case study, based on an investment project on exploration of an oil field under both deterministic and fuzzy conditions, is included to illustrate the use of these methods. Due to the complexity of the calculations involved, new MS-Excel financial functions are developed, by using Visual Basic for Applications. The main contribution of this paper is the development of a unifying approach to capital budgeting under uncertainty that emphasizes the strengths of the modified methods, while bypassing the individual conflicts and drawbacks of the conventional capital budgeting methods. Results confirm that the proposed solution has many advantages over other capital budgeting methods.

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1. Introduction

Over the last decades, several studies have been carried out to consider the issues of assessing investment projects under risk conditions, where all the information is usually treated as a probability distribution. In these cases, the probabilistic analysis is a powerful tool to solve capital budgeting problems (Huang, 2006; Uusitalo, Lehikoinen, Helle, & Myrberg, 2015).

However, this probabilistic view of an investment assessment does not provide easily implementable and manageable solutions for several real problems in a world under uncertain conditions. The uncertainty is intrinsically related to the process of assessing projects in which inaccurate or incomplete information is available and no probability distribution can be associated to the information.

Fuzzy set theory was introduced by Zadeh (1965) with the aim of modeling the imprecision and ambiguity of ordinary language.

In such cases, fuzzy numbers can be used to quantify inexact and uncertain information, in several technical and economic applications where the expert’s inaccurate and relatively vague knowledge is required to be accounted for in a quantitative manner.

Another tool for describing inaccuracy and uncertainty is the interval arithmetic (Moore, 1979; Moore & Lodwick, 2003; Moore, Kearfott, & Cloud, 2009). An interval number is the simplest way to represent lack of precision and inaccuracy, where there are no suppositions regarding the form of uncertainty between its higher and lower limits, and all types of uncertainties can be converted into an interval, having only the limits of such parameter.

The use of fuzzy sets and interval arithmetic theories associated with traditional capital budgeting methods has been widely investigated in the evaluation of investment projects under uncertainty conditions, and various fuzzy capital budgeting methods have been proposed in the last decades. Buckley (1987a, 1987b), Kaufmann (1986), Kaufmann and Gil-Aluja (1986, 1987), Kaufmann and Gupta (1988), Gutiérrez (1989), Ward (1985, 1989) papers were probably the first applications of the fuzzy and interval arithmetic theories in capital budgeting.

Ward (1985, 1989) developed a fuzzy present value analysis in which the cash flows are modeled as trapezoidal fuzzy num-
bers. Kaufmann (1986), Kaufmann and Gil-Aluja (1986, 1987) and Kaufmann and Gupta (1988) introduced the fuzzy present value formula in which the discount rates are modeled as triangular fuzzy numbers. Buckley (1987a) developed the fuzzy present value and fuzzy future value of fuzzy cash amounts using fuzzy interest rates over $n$ periods, where $n$ may be crisp or fuzzy. He has also presented a method of comparing fuzzy net cash flows in order to rank fuzzy investment alternatives. Buckley (1987b) considered the problem of ranking investment proposals, characterized by uncertain future cash flows, project duration and interest rates. Gutiérrez (1989) developed the fuzzy present value formula in which the cash flows and the discount rates are modeled as triangular fuzzy numbers and so make fuzzy cash flow analysis more manageable.

After those papers, several authors have contributed to refine the fuzzy capital budgeting theory (Banholzer, Sanches, Pamplona, & Montevetchi, 2005; Boussabaine & Elhag, 1999; Buckley, 1992; Carlsson & Fullér, 1999; Çetin & Kahraman, 1999; Chiu & Park, 1994, 1998; Dymowa, 2011; Gao, Zhao, & Ji, 2005; Gil-Aluja, 1997, 1999, 2004; Gil-Lafuente, 1990, 2001, 2005; Kahraman, Ruan, & Tolga, 2002; Kuchta, 2000; Mohamed & Mccowan, 2001; Sanches, Pamplona, & Montevetchi, 2005; Sergueiva & Hunter, 2004; Sheen, 2005; Terceño, Andrés, Barberá, & Lorentzana, 2003; Tolga, Demircan, & Kahraman, 2005; Tsao, 2005; Yao, Chen, & Lin, 2005), to name a few. Several authors emphasize the advantages of using fuzzy theory associated with traditional capital budgeting methods under uncertainty, and over a hundred articles can currently be found in the fuzzy financial literature. For example, Kahraman, Sari, Onar, and Özsayi (2016) developed engineering economy techniques under fuzziness to be employed in environmental problems. Buz (2015), Fathallahia and Najafi (2016), Kahraman, Çevik, and Özsayi (2016), Sari and Kahraman (2016), Zhao, Ke, and Chen (2016), and Appadoo (2014) developed fuzzy procedures based on the traditional net present value method.

Despite fuzzy capital budgeting having been studied extensively over the past 30 years and becoming academically popular in recent years, a literature review shows that the current fuzzy capital budgeting theory is fundamentally associated to the traditional deterministic methods of discounted cash flow and presents the same problems which can mislead the interpretation of the investment decisions. Such problems are related to, at least, three basic weaknesses of the traditional deterministic methods:

- **Multiple rates of return.** The internal rate of return results from the solution of an $n$-degree polynomial equation and, according to the Descartes’ Theorem, equations of this kind can include up to $n$ positive real roots. The existence of multiple rates, although mathematically correct, bears no relevant financial meaning to the capital investment decision-making process (Ross, Westerfield, & Jordan, 2008).

- **Implicit assumptions of the reinvestment rate.** The conflicting indications of projects obtained by using net present value (NPV) and internal rate of return (IRR) traditional methods originate, mainly, from the different implicit assumptions of reinvestment of the interim cash flows adopted in such methods (Brealey, Myers, & Allen, 2011). Such assumptions are that:

1. Emekçökgülu (2009), Sampaio Filho, Vellasco, and Tanscheit (2012), Guerra, Magni, and Stefanni (2012, 2014), Milanese, Pesce, and Alabi (2015). Shirinov et al. (2016) are exceptions to this trend. Emekçökgülu (2009) and Shirinov et al. (2016) papers are related with MIRR method, Sampaio Filho, Vellasco, & Tanscheit (2012) with MNPV method, Guerra et al. (2012, 2014) and Milanese et al. (2015), with IRR method. However, such proposals cannot be considered as a unified solution, as proposed in this paper, because they are based on only one modified capital budgeting method. Moreover, these proposals do not eliminate the main problems related to the traditional methods of investment appraisal.


Moreover, the major capital budgeting methods (net present value, internal rate of return, profitability index (PI), and payback (PB)) can lead to conflicting indications of acceptance or rejection of projects with non-conventional cash flows (Brealey et al., 2011). Largely discussed in the financial literature, these deterministic capital budgeting problems have been investigated and some authors have proposed alternative models to the conventional IRR and NPV methods, known as modified methods (Beaves, 1988, 1989, 1993, 1994, 2005; Biondi, 2006; McClure & Girma, 2004; Plath & Kennedy, 1994a, 1994b; Sampaio Filho, 2014; Shull, 1992, 1993, 1994; Vélez-Pareja, 2012). In practice, these modified methods are improved versions of the traditional IRR and NPV methods, eliminating problems regarding the financing and reinvestment rates, and different versions are known by different names and acronyms (Chandra, 2009).

Based on Sampaio Filho (2014), this paper extends the alternative models and proposes a unified solution in fuzzy capital budgeting that eliminates these problems of both deterministic and fuzzy techniques. In this sense, the rest of this paper is organized as follows. First, in Section 2, we introduce the general deterministic model of capital budgeting proposed by Sampaio Filho (2014). In Section 3 we introduce the fuzzy unified solution, the main contribution of this paper. Triangular fuzzy numbers are used to represent the uncertainties of the project variables: cash inflows, cash outflows and reinvestment, financing and risk-adjusted discount rates. In Section 4 we examine a comprehensive case study based on a real investment project on exploration of an oil field under both deterministic and fuzzy conditions. Finally, in Section 5 we present the conclusion of the paper.

### 2. The general deterministic model

The efforts of many researchers have generated, over the past six decades, a huge number of contributions aiming to tackle some of the problems mentioned in the previous section. Adjustments and modifications were introduced into the traditional discounted cash flow methods, generating new models and procedures referred to as “modified methods” or “adjusted methods” to assess investment projects.

Today there are many references in the financial literature to the modified internal rate of return (MIRR) and to the modified net present value (MNPV) methods, such as (Beaves, 1988, 1989, 1993, 1994, 2005; Casarotto Filho & Kopittke, 2010; Chandra, 2009; Crundwell, 2008; Fabozzi & Drake, 2009; Gapsinski, 2011; Kassai, Casanova, Santos, & Assaf Neto, 2007; McClure & Girma, 2004; Peterson & Fabozzi, 2002; Plath & Kennedy 1994a, 1994b; Vernimmen, Quiry, Le Fur, Salvi, & Dallochio, 2008; Vishwanath, 2007). These proposals correct the major problems of the traditional
IRR and NPV methods and make for more accurate investment decisions. Unfortunately, these MIRR and MNPV methods were established by applying different assumptions and may lead to different results.

Recently, Sampaio Filho (2014) has introduced a generalized solution that allows specifying the same assumptions for the major deterministic methods of capital budgeting. Based on Plath and Kennedy (1994a, 1994b), and McClure and Girma (2004) papers, Sampaio Filho (2014) has proposed the modified methods of the main capital budgeting (the modified internal rate of return (MIRR), the modified net present value (MNPV), The modified profitability index (MPI) and the modified total payback period (MTPB)) with the same assumptions. In one sense, these integrated methods are not new; they are a generalized reformulation of the various MIRR and MNPV proposals. This generalized solution eliminates the major problems of the traditional methods of capital budgeting: the possibility of multiple rates of return of the IRR method; the implicit assumptions of the reinvestment rates of the IRR and NPV methods; and the anomalous behavior of the NPV method. In addition, all proposed methods (MNPV, MIRR, MPI and MTPB) lead to same consistent indications of acceptance or rejection of projects with non-conventional cash flows.

The MIRR, MNPV, MPI and MTPB methods proposed by Sampaio Filho (2014) were established from the unified model represented in Fig. 1. According to this diagram, the intermediate negative cash flows (NCF) are discounted down to period zero at the financing rates, obtaining the Present Value (PV), while the intermediate positive cash flows (PCF) are capitalized up to period n at the reinvestment rate, and then obtaining the Terminal Value (TV). The terminal value (TV) and the present value (PV), which are basic indexes for all modified indicators, are obtained from Eqs. (1) and (2):

\[ TV = \sum_{t=1}^{n} PCF_t (1 + k_{rad})^{n-t} \]  

\[ PV = \sum_{t=0}^{n} NCF_t \frac{1}{(1 + k_{wacc})^t} \]  

where \( PCF_t \) is the positive cash flows by the end of period \( t \); \( n \) is the project’s lifespan; \( k_{re} \) is the reinvestment rate; \( NCF_t \) is the negative cash flows by the end of period \( t \); \( k_{wacc} \) is the financing rate.

From these two unique values (TV and PV), deterministic indicators of MIRR, MNPV, MPI and MTPB can be established, represented by Eqs. (3)–(6), respectively:

\[ MNPV = \frac{TV}{(1 + k_{rad})^n} - PV \]  

\[ MIRR = \left(\frac{TV}{PV}\right)^{1/n} - 1 \]  

\[ MPI = \frac{TV(1 + k_{rad})^{-n}}{PV} \]  

\[ MTPB = \frac{PV}{TV(1 + k_{rad})^{-n}x^n} \]  

where \( k_{rad} \) is the risk-adjusted discount rate.

3. The fuzzy unified solution

Based on the modified capital budgeting model introduced by Sampaio Filho (2014), this paper proposes a fuzzy unified solution that bypasses the problems of fuzzy capital budgeting methods in different investment projects with different settings of rates and cash flows. Triangular fuzzy numbers are used to represent uncertainty of cash flows and of reinvestment, financing and risk-adjusted discount rates.

Let us assume that cash flows, financing rate, reinvestment rate and risk-adjusted discount rate associated to an investment project under uncertainty conditions can be represented by triangular fuzzy numbers. These fuzzy numbers assume the form presented in Fig. 2, where an hypothetic \( \alpha \)-cut is also shown.

A L-R fuzzy number can be represented as:

\[ A^\alpha = [a^{l(\alpha)}, a^{r(\alpha)}] \]  

where, \( l(\alpha) \) represents the left line of the fuzzy number, and \( r(\alpha) \), the right line of it.

The \( \alpha \)-cut notation is also frequently used to represent fuzzy numbers. In this case:

\[ \frac{a^{l(\alpha)} - a_1}{a_2 - a_1} = \alpha, \frac{a_3 - a^{r(\alpha)}}{a_3 - a_2} = \alpha \]  

Such as:

\[ a^{l(\alpha)} = a_1 + (a_2 - a_1)\alpha, a^{r(\alpha)} = a_3 + (a_2 - a_3)\alpha \]  

Fig. 1. Capital budgeting unified model and resulting cash flow.

Fig. 2. Triangular fuzzy number.
Thus,
\[ A^\alpha = [a^1(\alpha), a^2(\alpha)] = [a_1 + (a_2 - a_1)\alpha, a_2 + (a_2 - a_1)\alpha] \quad \alpha \in [0, 1] \] (10)

From Eq. (10), mathematical notation regarding triangular fuzzy numbers corresponding to cash flows (\(CF_t\)), under both reinvestment (\(k_{rr}\)) and financing (\(k_{wacc}\)) rates and risk-adjusted discount rate (\(k_{radr}\)) can be established according to Eq. (11)-(14).

\[ CF^\alpha = [CF_1 + (CF_2 - CF_1)\alpha, CF_3 + (CF_2 - CF_1)\alpha] \] (11)

\[ k_{rr}^\alpha = [k_{rr1} + (k_{rr2} - k_{rr1})\alpha, k_{rr3} + (k_{rr2} - k_{rr3})\alpha] \] (12)

\[ k_{wacc}^\alpha = [k_{wacc1} + (k_{wacc2} - k_{wacc1})\alpha, k_{wacc3} + (k_{wacc2} - k_{wacc3})\alpha] \] (13)

\[ k_{radr}^\alpha = [k_{radr1} + (k_{radr2} - k_{radr1})\alpha, k_{radr3} + (k_{radr2} - k_{radr3})\alpha] \] (14)

3.1. Obtaining the fuzzy terminal value (\(TV^\alpha\))

Developed from Eqs. (1), (11) and (12), the fuzzy terminal value (\(TV^\alpha\)) is the sum of all fuzzy positive cash flows (\(PCF^\alpha\)) capitalized under fuzzy reinvestment rate (\(k_{rr}^\alpha\));

\[ TV^\alpha = PCF_1^\alpha (1 + k_{rr}^\alpha)^{n-1} + PCF_2^\alpha (1 + k_{rr}^\alpha)^{n-2} + \ldots + PCF_n^\alpha (1 + k_{rr}^\alpha)^0 \quad PCF_n^\alpha > 0 \] (15)

According to Moore et al. (2009), each interval defining a fuzzy number shall comply with the following conditions:

\[
\begin{align*}
[a^1, a^2] + [b^1, b^2] &= [a^1 + b^1, a^2 + b^2] \\
- [a^1, a^2] &= [-a^2, -a^1] \\
[a^1, a^2] - [b^1, b^2] &= [a^1 - b^1, a^2 - b^2] \\
[a^1, a^2]^{-1} &= (1/|a^1, a^2|) = [1/a^2, 1/a^1] \\
[a^1, a^2] \cdot [b^1, b^2] &= [a^1b^1, a^2b^2] \quad a^1 \geq 0 b^1 \geq 0 \\
[a^1, a^2] / [b^1, b^2] &= [a^1/b^1, a^2/b^2] \quad a^1 > 0 b^2 > 0 \\
[a^1, a^2] + [b^1, b^2] &= [a^1 + b^1, a^2 + b^2] \quad a^1 < 0 b^1 < 0 \\
[a^1, a^2] / [b^1, b^2] &= [a^1/b^1, a^2/b^2] \quad a^1 > 0 b^2 < 0 \hspace{1cm} [1, 1] + [b^1, b^2] = [1/b^1, 1/b^2] \\
\sqrt{[a^1, a^2]} &= \sqrt{a^1, a^2} \\
[a^1, a^2]^n &= [a_1^n, a_2^n] \quad a^1 \geq 0 \\
[a^1, a^2]^0 &= [1, 1]
\end{align*}
\] (16)

By applying the conditions stated in (16) in Eq. (15):

\[ TV^\alpha = [PCF_1^{\alpha(1)}x(1 + k_{rr}^{\alpha(1)})^{n-1}, PCF_1^{\alpha(1)}x(1 + k_{rr}^{\alpha(1)})^{n-2}] + [PCF_2^{\alpha(1)}x(1 + k_{rr}^{\alpha(1)})^{n-2}, PCF_2^{\alpha(1)}x(1 + k_{rr}^{\alpha(1)})^{n-2}] + \ldots + [PCF_n^{\alpha(1)}, PCF_n^{\alpha(1)}] \] (17)

where:

\[ TV^\alpha = PCF_1^{\alpha(1)}x(1 + k_{rr}^{\alpha(1)})^{n-1} + PCF_1^{\alpha(1)}x(1 + k_{rr}^{\alpha(1)})^{n-2} + \ldots + PCF_n^{\alpha(1)} \] (18)

\[ TV^\alpha = PCF_1^{\alpha(1)}x(1 + k_{rr}^{\alpha(1)})^{n-1} + PCF_1^{\alpha(1)}x(1 + k_{rr}^{\alpha(1)})^{n-2} + \ldots + PCF_n^{\alpha(1)} \] (19)

3.2. Obtaining the fuzzy present value (\(PV^\alpha\))

Developed from Eq. (2) and Eq. (11) to (14), the fuzzy terminal value (\(PV^\alpha\)) is the sum of all fuzzy negative cash flows (\(NCF^\alpha\)) discounted under fuzzy financing rate (\(k_{wacc}^\alpha\));

\[ PV^\alpha = NCF_1^{\alpha(1)} + \frac{NCF_1^{\alpha(1)}}{1 + k_{wacc}^{\alpha(1)}} + \frac{NCF_2^{\alpha(1)}}{1 + k_{wacc}^{\alpha(1)}}^2 + \ldots + \frac{NCF_n^{\alpha(1)}}{1 + k_{wacc}^{\alpha(1)}}^n \quad NCF_n^{\alpha(1)} < 0 \] (20)

By applying the conditions stated in (16) in Eq. (20);

\[ PV^\alpha = [NCF_1^{\alpha(1)}, NCF_1^{\alpha(1)}] + [\frac{NCF_1^{\alpha(1)}}{1 + k_{wacc}^{\alpha(1)}}^{\alpha(1)}, \frac{NCF_1^{\alpha(1)}}{1 + k_{wacc}^{\alpha(1)}}^{\alpha(1)}] + \ldots + [\frac{NCF_n^{\alpha(1)}}{1 + k_{wacc}^{\alpha(1)}}^{\alpha(1)}, \frac{NCF_n^{\alpha(1)}}{1 + k_{wacc}^{\alpha(1)}}^{\alpha(1)}] \] (21)

where:

\[ PV^\alpha = NCF_1^{\alpha(1)} + \frac{NCF_1^{\alpha(1)}}{1 + k_{wacc}^{\alpha(1)}}^{\alpha(1)} + \frac{NCF_2^{\alpha(1)}}{1 + k_{wacc}^{\alpha(1)}}^{\alpha(1)} + \ldots + \frac{NCF_n^{\alpha(1)}}{1 + k_{wacc}^{\alpha(1)}}^{\alpha(1)} \] (22)

\[ PV^\alpha = NCF_1^{\alpha(1)} + \frac{NCF_1^{\alpha(1)}}{1 + k_{wacc}^{\alpha(1)}}^{\alpha(1)} + \frac{NCF_2^{\alpha(1)}}{1 + k_{wacc}^{\alpha(1)}}^{\alpha(1)} + \ldots + \frac{NCF_n^{\alpha(1)}}{1 + k_{wacc}^{\alpha(1)}}^{\alpha(1)} \] (23)

3.3. Obtaining the fuzzy modified net present value (\(MNPV^\alpha\))

From Eq. (3), the fuzzy MNPV can be represented as:

\[ MNPV^\alpha = \frac{TV^\alpha}{(1 + k_{radr}^{\alpha})^n} + PV^\alpha \\ TV^\alpha > 0 \text{ and } PV^\alpha < 0 \] (24)

By applying the conditions stated in (16) on Eq. (24);

\[ MNPV^\alpha = \left[ \frac{TV^{\alpha(1)}}{1 + k_{radr}^{\alpha(1)}}^{\alpha(1)}, \frac{TV^{\alpha(1)}}{1 + k_{radr}^{\alpha(1)}}^{\alpha(1)} \right] + [PV^{\alpha(1)}, PV^{\alpha(1)}] \] (25)

where:

\[ MNPV^{\alpha(1)} = \frac{TV^{\alpha(1)}}{1 + k_{radr}^{\alpha(1)}}^{\alpha(1)} + PV^{\alpha(1)} \] (26)

\[ MNPV^{\alpha(1)} = \frac{TV^{\alpha(1)}}{1 + k_{radr}^{\alpha(1)}}^{\alpha(1)} + PV^{\alpha(1)} \] (27)

3.4. Obtaining the fuzzy modified internal rate of return (\(MIRR^\alpha\))

From Eq. (4), the fuzzy MIRR can be represented as:

\[ MIRR^\alpha = \left[ \frac{TV^{\alpha(1)}}{PV^{\alpha(1)}} \right]^{1/m} - 1 \\ TV^\alpha > 0 \text{ and } PV^\alpha < 0 \] (28)

By applying the conditions stated in (16) in Eq. (28);

\[ MIRR^\alpha = \left[ \frac{TV^{\alpha(1)}}{PV^{\alpha(1)}} \right]^{1/m} - 1 \] (29)
where:

\[
\text{MIRR}^{(a)} = \left[ \frac{-TV^{(a)}}{PV^{(a)}} \right]^{1/m} - 1
\]  

(30)

\[
\text{MIRR}^{(a)} = \left[ \frac{-TV^{(a)}}{PV^{(a)}} \right]^{1/m} - 1
\]  

(31)

3.5. Obtaining the fuzzy modified profitability index (MPI\textsuperscript{a})

From Eq. (5), the fuzzy MPI can be represented as:

\[
\text{MPI}^{a} = -\frac{TV^{a}}{(1 + k_{rad}^{a})^n}/PV^{a} \quad TV^{a} > 0 \text{ and } PV^{a} < 0
\]  

(32)

By applying the conditions stated in (16) in Eq. (32):

\[
\text{MPI}^{a} = \left[ -\frac{TV^{(a)}}{(1 + k_{rad}^{(a)})^n} - \frac{TV^{(a)}}{(1 + k_{rad}^{(a)})^n} \right] / [PV^{(a)}, PV^{(a)}]
\]  

(33)

where:

\[
\text{MPI}^{(a)} = \left[ -\frac{TV^{(a)}}{(1 + k_{rad}^{(a)})^n} \right] / [PV^{(a)}]
\]  

(34)

\[
\text{MPI}^{(a)} = \left[ -\frac{TV^{(a)}}{(1 + k_{rad}^{(a)})^n} \right] / [PV^{(a)}]
\]  

(35)

3.6. Obtaining the fuzzy modified total payback period (MTPB\textsuperscript{a})

From Eq. (6), the fuzzy MTPB can be represented as:

\[
\text{MTPB}^{a} = -\frac{TV^{a}}{(1 + k_{rad}^{a})^n} \times n \quad TV^{a} > 0 \text{ and } PV^{a} < 0
\]  

(36)

By applying the conditions stated in (16) in Eq. (36), the MTPB\textsuperscript{a} is achieved by solving the following equation:

\[
\text{MTPB}^{a} = \left[ -PV^{(a)}, -PV^{(a)} \right] / \left[ \frac{TV^{(a)}}{(1 + k_{rad}^{(a)})^n}, \frac{TV^{(a)}}{(1 + k_{rad}^{(a)})^n} \right] \times n
\]  

(37)

where:

\[
\text{MTPB}^{(a)} = -PV^{(a)} / \left[ \frac{TV^{(a)}}{(1 + k_{rad}^{(a)})^n} \right] \times n
\]  

(38)

\[
\text{MTPB}^{(a)} = -PV^{(a)} / \left[ \frac{TV^{(a)}}{(1 + k_{rad}^{(a)})^n} \right] \times n
\]  

(39)

Decisions about investment must consider three different dimensions: absolute return, relative return and liquidity of a project. There is no evaluation measure that may capture all those dimensions at the same time. Many deterministic and fuzzy solutions have been proposed in the last decades to deal with capital budgeting, but the unified approach proposed here distinguishes itself by being a generalized one regarding absolute return (MNPV), relative return (MIRR and MPI), and liquidity (MTPB). All of these lead to consistent indications of acceptance or rejection of an investment project. In the next section the proposed model is applied to an actual case: analysis of an investment project in the petroleum area, where the consistent indications will be confirmed.

4. Case study

The problem consists of accepting or rejecting a project for the exploration and production of oil fields in a Brazilian company. The investments involve values of around US$ 3,800 million—a high amount for any large-scale company. As there are no concurrent alternative projects, the decision to be made is simply whether to accept or reject the project. The simple determination of the presence of a positive net value would justify the investment. However, due to the characteristics of projects in the oil area, the exclusive use of the traditional NPV method would not provide sufficient reliability for the decision to be made, and would require the adoption of procedures to treat the uncertainties, as well as the establishment of the possibility of failure of the project.

The uncertainties associated with the enterprise, related to the political and economic situation of the country, led the company to seek alternative methods for evaluating the viability of the project. A suggestion was made to adopt methodologies that can more adequately represent the market conditions and uncertainties of the variables involved.

The evaluation of enterprises in the oil sector requires input variables such as future prices of products, prediction of production during the project’s lifespan, initial investments, operational costs, abandonment costs, and interest rates. Many uncertainties are present in the estimates of these variables, some of which are easily quantifiable, and can be determined by company specialists.

4.1. The investment project

A project for the development of oil fields is proposed for the year 2010. The total investments required to explore the oil wells are of US$ 3766.9 million, of which US$ 1324.8 million are for drilling; US$ 1687.5 million are for production installations; and US$ 754.6 million for collection and outflow, effectuated between 2010 and 2017. These values are subject to linear depreciation with annual recovery as of 2014. Abandonment costs are also predicted to be spent in 2033, in the value of US$ 450 million. The annual oil production estimates, in addition to fixed annual operational costs, variable annual costs, commercialization rates (US$/bbl.), royalties, annual inflation, and income tax aliquid. The company analysis of the investment project must be based on the evaluation of the net cash flow displayed in Table 1.

Due to the practical limitations for the precise establishment of the exact moments at which the costs and revenues occur, the timeline was divided into years and the same position was established for the annual monetary values, at the end of each period. Currently, the average weighted cost of capital and investment opportunity rate adopted by the company are 10.0% and 13.0% per year, respectively. The risk-adjusted discount rates are 7.0%, 10.0%, and 13.0% per year, for low-, medium-, and high-risk projects, respectively. It can be noted in Table 1 that the investment project presents a non-conventional cash flow: a negative cash flow in the zero period (2010); followed by four negative cash flows from 2011 to 2014; eighteen positive cash flows from 2015 to 2032; and a negative cash flow corresponding to the cost of abandoning the project, during the last year of the project (2033), which lasts twenty-three years.

\footnote{Abandonment costs are a fundamental component of any financial viability analysis of projects related to exploring and producing oil fields. The abandonment costs are extremely elevated and correspond, mainly, to the inherent dismantling, removal, and restoratation of the areas involved in the production of oil and gas. These costs exceed, in some cases, the investments in infrastructure and installation of machines and equipment required for production.}

\cite{SampaioFilho2014}
Table 1
Net cash flow of the oil field development project.

<table>
<thead>
<tr>
<th>Year</th>
<th>Net cash flow (US$MM)</th>
</tr>
</thead>
<tbody>
<tr>
<td>2010</td>
<td>$32.05</td>
</tr>
<tr>
<td>2011</td>
<td>$(429.4)</td>
</tr>
<tr>
<td>2012</td>
<td>$(958.6)</td>
</tr>
<tr>
<td>2013</td>
<td>$(1051.7)</td>
</tr>
<tr>
<td>2014</td>
<td>$(341.9)</td>
</tr>
<tr>
<td>2015</td>
<td>$1158.8</td>
</tr>
<tr>
<td>2016</td>
<td>$1409.1</td>
</tr>
<tr>
<td>2017</td>
<td>$1041.5</td>
</tr>
<tr>
<td>2018</td>
<td>$722.8</td>
</tr>
<tr>
<td>2019</td>
<td>$541.2</td>
</tr>
</tbody>
</table>

Table 2
Traditional deterministic indices of the investment project.

<table>
<thead>
<tr>
<th>Traditional Method</th>
<th>Risk-adjusted discount rate ($k_{adj}$)</th>
<th>US$ MM</th>
</tr>
</thead>
<tbody>
<tr>
<td>NPV</td>
<td>7%</td>
<td>$1,563.95</td>
</tr>
<tr>
<td></td>
<td>10%</td>
<td>$972.95</td>
</tr>
<tr>
<td></td>
<td>13%</td>
<td>$532.05</td>
</tr>
<tr>
<td>IRR</td>
<td>18.4% and -20.6%</td>
<td></td>
</tr>
<tr>
<td>PI</td>
<td>1.63</td>
<td></td>
</tr>
<tr>
<td>TPB</td>
<td>14.11</td>
<td></td>
</tr>
</tbody>
</table>

Similarly, given the project lifespan (23 years), with the maximum acceptable time for the investment return, the indices from the total payback period indicate the acceptance of the project, since all indices are less than 23 years (Low Risk TPB < Medium Risk TPB < High Risk TPB < 23 years).

The use of the NPV method assumes that the cash flows and interest rates are known for certain and that the uncertainties of these variables are included in the choice of the risk-adjusted discount rate. However, these uncertainties may threaten the viability of the project or provide the impression of a higher NPV.

It can be noted that the risk classification of the project is a fundamental parameter for the adequate evaluation of the viability of the project. In this case, the classification of the project as low- or medium-risk, and the use of the respective risk-adjusted discount rates would result in oversized values of the NPV (1563.95/532.05 – 1 = 193.35% and 972.95/532.05 – 1 = 82.87%, respectively) when compared to the NPV of the high-risk project.

4.2. Project analysis under deterministic conditions

In the analysis of the investment project under deterministic conditions, traditional methods (NPV, IRR, PI, and TPB), and modified methods (MNPV, MTPB, MPI, and MTPB) are used.

4.2.1. Application of traditional methods

According to Brealey et al. (2011), Ross et al. (2008), and Kassai et al. (2007), Table 2 presents values of the NPV, IRR, PI, and TPB traditional indices, calculated for the risk-adjusted discount rates of 7%, 10%, and 13% per year, used by the company in low-, medium-, and high-risk investment projects.

The project cash flow is non-conventional, presenting two sign variations and, consequently, two internal return rates, which makes the use of the IRR inadequate to the analysis of this project. Fig. 3 shows the variation of the NPV for various discount rates, displaying two internal return rates: 18.4% and -20.6% per year. According to the values obtained through the traditional NPV method, the investment project must be accepted, regardless of the risk associated with the investment project, since all NPV indices are positive (Low Risk NPV > Medium Risk NPV > High Risk NPV > 0).

The indices for the profitability index (PI) method indicate that the project must be accepted, since all of them are greater than 1 (one) (Low Risk PI > Medium Risk PI > High Risk PI > 1).

Similarly, given the project lifespan (23 years), with the maximum acceptable time for the investment return, the indices from the total payback period indicate the acceptance of the project, since all indices are less than 23 years (Low Risk TPB < Medium Risk TPB < High Risk TPB < 23 years).

The use of the NPV method assumes that the cash flows and interest rates are known for certain and that the uncertainties of these variables are included in the choice of the risk-adjusted discount rate. However, these uncertainties may threaten the viability of the project or provide the impression of a higher NPV.

It can be noted that the risk classification of the project is a fundamental parameter for the adequate evaluation of the viability of the project. In this case, the classification of the project as low- or medium-risk, and the use of the respective risk-adjusted discount rates would result in oversized values of the NPV (1563.95/532.05 – 1 = 193.35% and 972.95/532.05 – 1 = 82.87%, respectively) when compared to the NPV of the high-risk project.

4.2.2. Application of modified methods

MNPV, MIRR, MPI and MTPB indices are obtained based on Eqs. (3)–(6), where the terminal value (TV) and present value (PV) are obtained through Eqs. (1) and (2). Table 3 presents the values of these indices calculated for reinvestment rates varying from 10% to 15% per year, and setting the risk-adjusted discount rate to 13% per year, a rate used by the company to evaluate high-risk investment projects.

The use of reinvestment, financing and risk-adjusted discount rates adopted by the organization for high-risk projects would lead to the acceptance of the project (MNPV = $371.88 > 0, MIRR = 13.74% > k_{adj} = 13% per year, MPI = 1.16 > 1 and MTPB < 23 years, project lifespan) based on the adoption of any of the modified indices.

When comparing the results of Tables 2 and 3, it can be observed that the indicator of the traditional NPV is 43.1% = $532.05/371.88 – 1, overestimated in relation to the modified NPV. Similarly, the PI indicator is 7.8% = (1.25/1.16 – 1), overestimated in relation to the MPI, and the TPB is 7.0% (18.40/19.78 – 1), overestimated in relation to the MTPB.

It can be noted in Table 3 that, when project intermediary input cash flows cannot be reinvested by the company at rates

\[ \text{\ldots} \]

\[ \text{\ldots} \]
equal to or higher than approximately 12% per year, the MNPV is negative, the MIRR is lower than the $k_{\text{radr}}$ (13% per year), the MPI is less than 1, and the MTPB is higher than the project lifespan (23 years)—which would result in the joint rejection of the project (or in the possibility of its failure in the future).

### Table 3

<table>
<thead>
<tr>
<th>Reinvestment Rate</th>
<th>MNPV</th>
<th>MIRR</th>
<th>MPI</th>
<th>MTPB</th>
</tr>
</thead>
<tbody>
<tr>
<td>15.0%</td>
<td>$1.209$</td>
<td>15.10%</td>
<td>1.53</td>
<td>15.05</td>
</tr>
<tr>
<td>14.0%</td>
<td>$762.87$</td>
<td>14.42%</td>
<td>1.33</td>
<td>17.25</td>
</tr>
<tr>
<td>13.0%</td>
<td>$371.88$</td>
<td>13.74%</td>
<td>1.16</td>
<td>19.78</td>
</tr>
<tr>
<td>12.0%</td>
<td>$29.63$</td>
<td>13.06%</td>
<td>1.01</td>
<td>22.71</td>
</tr>
<tr>
<td>11.0%</td>
<td>$-269.73$</td>
<td>12.39%</td>
<td>0.88</td>
<td>26.08</td>
</tr>
<tr>
<td>10.0%</td>
<td>$-531.36$</td>
<td>11.71%</td>
<td>0.77</td>
<td>29.96</td>
</tr>
</tbody>
</table>

**Risk-adjusted discount rate ($k_{\text{radr}}$) 13.0%**

**Weighted average cost of capital ($k_{\text{wacc}}$) 10.0%**

#### 4.2.3. NPV vs. MNPV profiles

Fig. 4 graphically represents the relationship between the NPV and the MNPV of the investment project, when the risk-adjusted discount rate varies from 5% to 20% per year. The graphic analysis of the profile of the NPV vs. MNPV of the project shows that the NPV is equal to zero when the discount rate is equal to the IRR (18.37% per year) of the project. On the other hand, the MNPV is equal to zero when the discount rate is equal to the MIRR (11.71% per year). The profile of the MNPV is established assuming the hypothesis of a constant reinvestment rate equal to the $k_{\text{wacc}}$, and that the discount rate varies as in the profile of the NPV.

It can be noted that, when the $k_{\text{radr}}$ of the project is less than the capital cost, the calculation of the NPV underestimates the result of the project. Fig. 4 shows that this difference increases with the increase of the difference between the $k_{\text{radr}}$ of the project and the capital cost. According to McClure and Girma (2004), this occurs because the calculation of the NPV implicitly assumes that the intermediary cash flows of the project are reinvested in a lower $k_{\text{radr}}$ than the capital cost. This underestimation of the NPV may lead to the rejection of low-risk projects that could add value to the company shareholders. Also according to the authors cited, for a company that aims to maximize the wealth of the shareholders, the reinvestment of positive intermediary cash flows at a rate below the $k_{\text{wacc}}$ of the organization is inconsistent with the objective of maximizing their wealth.

On the other hand, when the $k_{\text{radr}}$ is greater than the company’s $k_{\text{wacc}}$, the NPV overestimates the result of the project. This overestimation leads to the acceptance of high-risk projects that could decrease value for the shareholders of the firm. Again, this overestimation occurs because the calculation of the NPV assumes that the intermediary cash flows of the high-risk project will be reinvested in other similar high-risk project. In addition, the deviations of the results are not symmetrical and vary according to the pattern of cash flows of the project, the degree of risk of the project being analyzed, the reinvestment rate, and the financing rate. Although the deviations are greater for the high-risk projects and lower for the low-risk projects, the results still lead to the acceptance of high-risk projects and rejection of low-risk projects. Finally, it can be noted that the NPV and MNPV are equal when the discount, financing, and reinvestment rates are the same. When the discount rate is different from the reinvestment rate,
the MNPV provides a more realistic decision-making tool because it considers the available reinvestment opportunities for the firm.

4.3. Project analysis under fuzzy conditions

In the items above, the viability analyses of the enterprise were performed based on the deterministic cash flows and discount rates. However, this procedure is merely a simplification, as future events such as revenue, costs, and discount rates are not fully known a priori. In this regard, the analyses assume uncertainty.

For this purpose, it is acknowledged that the estimates of the free cash flow, reinvestment rates, and financing rates of the company—and those for the risk-adjusted discount rates for the for low, medium, and high-risk projects—were established as triangular fuzzy numbers, based on the values of Table 1 with a margin of ±10% of uncertainty (Tables 4 and 5).

4.3.1. Application of fuzzy NPV traditional method

As previously stated in this article, the mathematical notations for triangular fuzzy numbers corresponding to the cash flows (CFs), reinvestment rates \(k_r\), financing rates \(k_{necc}\), and risk-adjusted discount rates \(k_{radr}\) can be established, according to Eqs. (11)–(14), as follows:

\[CF_0 = \left[-33.5 + (-30.3 + 33.5)\alpha, -27.4 - (27.4 + 30.5)\alpha\right]\]

\[CF_1 = \left[-427.4 + (-429.4 + 427.4)\alpha, -386.5 - (-386.5 + 429.4)\alpha\right]\]

\[\ldots\]

\[CF_{23} = \left[-495 + (-450 + 495)\alpha, -405 - (-405 + 450)\alpha\right]\]

For \(\alpha = 0:\)

\[CF_1^{radr} = \left[-33.5 + 3\times 0, -27.4 - 3.1\times 0\right] = \left[-33.5, -27.4\right]\]

\[CF_1^{radr} = \left[-472.4 + 43\times 0, -386.5 - 42.9\times 0\right] = \left[-472.4, -386.5\right]\]

\[\ldots\]

\[CF_{23}^{radr} = \left[-495 + 45\times 0, -405 - 45\times 0\right] = \left[-495, -405\right]\]

The values of the fuzzy cash flows of \(CF_{0}^{radr}\) to \(CF_{23}^{radr}\) are shown in Table 6 for different \(\alpha\) values varying from 0 to 1.

4.3.1.2. Calculation of fuzzy risk-adjusted discount rates \(k_{radr}^{(\alpha)}\).

Using similar procedures to those adopted in the previous item, the different values of \(k_{radr}^{(\alpha)}\) are calculated and displayed in Table 7.

4.3.1.3. Calculation of fuzzy net present value (NPV\(^{(\alpha)}\)).

The fuzzy net present value (fuzzy NPV) is the most discussed method in the fuzzy capital budgeting literature. Ward (1985) describes an application of the fuzzy set theory to engineering economy in which the cash flows are modeled as fuzzy numbers rather than crisp ones. In the example used by Ward (1985), cash flows are assumed to be flat fuzzy filter function numbers (4F numbers) having symmetrical triangular membership functions. According to this author, if the cash flows are fuzzy numbers, extended multiplication and addition can be used to discount and sum them in order to obtain their fuzzy present value. On the other hand, Gutiérrez (1989) and Chiu and Park (1994, 1998) have used fuzzy theory as a tool in determining the net present value of an investment project, where the cash flows and the rates of discount are viewed as fuzzy numbers.

According to Chiu and Park (1994, 1998), Gutiérrez (1989), and Ward (1985) papers, the fuzzy net present value (NPV\(^{(\alpha)}\)) shall be calculated as follows:

\[NPV^{(\alpha=0)} = CF_0^{(\alpha=0)} + \frac{CF_1^{(\alpha=0)}}{1 + k_{radr}^{(\alpha=0)}} + \ldots + \frac{CF_{23}^{(\alpha=0)}}{(1 + k_{radr}^{(\alpha=0)})^{23}}\]

\[= -33.5 + \frac{-472.4}{1 + 0.117} + \ldots + \frac{37.4}{(1 + 0.143)^{22}} - 495.0\]

\[= -33.5 + 2.0 - 38.8 = -213.75\]

One should be very careful when calculating the fuzzy NPV, because the cash flows of the project for \(n > 0\) are not all positive fuzzy numbers, as it can be seen in Table 6 (\(CF_1^{(\alpha)}, CF_2^{(\alpha)}, CF_3^{(\alpha)}, CF_4^{(\alpha)}, CF_5^{(\alpha)}, CF_6^{(\alpha)} < 0\)). In this case, the application of the interval arithmetic properties in the fuzzy NPV expression must obey the following conditions:

\[\alpha = 0:\]

\[NPV^{(\alpha=0)} = CF_0^{(\alpha=0)} + \frac{CF_1^{(\alpha=0)}}{1 + k_{radr}^{(\alpha=0)}} + \ldots + \frac{CF_{23}^{(\alpha=0)}}{(1 + k_{radr}^{(\alpha=0)})^{23}}\]

\[= -27.4 + \frac{-386.5}{1 + 0.143} + \ldots + \frac{45.7}{(1 + 0.143)^{22}} - 405.0\]

\[= -27.4 - 338.1 + \ldots + 4.0 - 18.7 = 1,331.51\]
Table 6
Fuzzy cash flow values of the investment project.

<table>
<thead>
<tr>
<th>(FC^n_a)</th>
<th>(FC^b_a)</th>
<th>(FC^c_a)</th>
<th>(FC^d_a)</th>
<th>(FC^e_a)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(FC^{l(a)})</td>
<td>(FC^{l(b)})</td>
<td>(FC^{l(c)})</td>
<td>(FC^{l(d)})</td>
<td>(FC^{l(e)})</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>(FC^a)</th>
<th>(FC^b)</th>
<th>(FC^c)</th>
<th>(FC^d)</th>
<th>(FC^e)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(FC^{l(a)})</td>
<td>(FC^{l(b)})</td>
<td>(FC^{l(c)})</td>
<td>(FC^{l(d)})</td>
<td>(FC^{l(e)})</td>
</tr>
</tbody>
</table>

\(\alpha\) values shown in Table 8. Table 8 shows \(\alpha\) values varying from 0 to 1.

A graphical representation of the \(NPV^a\) is shown in Fig. 5. The most promising fuzzy net present value is $532.05. The highest and lowest values of the fuzzy \(NPV\) are $1313.51 and $-213.75, respectively. The area with negative fuzzy \(NPV\) indicates that there is a small possibility that the project will not be justified as acceptable. The possibility can be calculated as approximately 4%, dividing the negative area by the total area.

4.3.2. Application of fuzzy modified methods
The calculations of fuzzy \(MNVP\), \(MIKR\), \(MPI\) and \(MTPB\) values are divided into five steps:

i. For the positive cash flows:

\[|a^d, a'\| | b, b'| = |a^d, a'/ b, b'| a^d > 0 b > 0\]

ii. For the negative cash flows:

\[|a^d, a'| | b, b'| = |a^d, a'/ b, b'| a^d < 0 b > 0\]

The fuzzy \(NPV\) values are shown in Table 8, for \(\alpha\) values varying from 0 to 1.

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ii. For the negative cash flows:

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4.3.2.1. Calculation of fuzzy cash flows ($C_{F}{}^\alpha{}$). The procedures for calculating the fuzzy cash flows for the application of the proposed methods are identical to those used previously for the calculation of the fuzzy cash flows for the fuzzy NPV. The values of $C_{F}{}^\alpha{}$ through $C_{F}{}^{23}{}^\alpha{}$ are the same as those presented in Table 6, for different values of $\alpha$ varying from 0 to 1.

4.3.2.2. Calculation of fuzzy rates ($k_{n}{}^\alpha$, $k_{\text{wacc}}{}^\alpha$ and $k_{\text{radr}}{}^\alpha$). Using similar procedures to those adopted in the previous item to calculate the fuzzy risk-adjusted discount rate, the different values of $k_{n}{}^\alpha$, $k_{\text{wacc}}{}^\alpha$, and $k_{\text{radr}}{}^\alpha$ are calculated and shown in Table 9:

4.3.2.3. Calculation of fuzzy terminal values ($TV_{\alpha}$). According to Eq. (17), the fuzzy terminal fuzzy is the sum of all fuzzy positive cash flows ($PC_{F}{}^\alpha$) capitalized at the fuzzy reinvestment rate ($k_{rr}{}^\alpha$) until the last period of the project ($n=23$):

$$TV_{\alpha} = [TV_{1}{}^{\alpha} + TV_{2}{}^{\alpha} + \ldots + TV_{n}{}^{\alpha}]$$

$$= [PC_{F}{}_{1}{}^{\alpha} (1 + k_{rr}{}_{1}{}^{\alpha})^{18} + PC_{F}{}_{2}{}^{\alpha} (1 + k_{rr}{}_{2}{}^{\alpha})^{17} + \ldots + PC_{F}{}_{23}{}^{\alpha} (1 + k_{rr}{}_{23}{}^{\alpha})^{1}]$$

where:

$$TV_{1}{}^{\alpha} = PC_{F}{}_{1}{}^{\alpha} (1 + k_{rr}{}_{1}{}^{\alpha})^{18} + PC_{F}{}_{2}{}^{\alpha} (1 + k_{rr}{}_{2}{}^{\alpha})^{17} + \ldots + PC_{F}{}_{23}{}^{\alpha} (1 + k_{rr}{}_{23}{}^{\alpha})^{1}$$

$$TV_{2}{}^{\alpha} = PC_{F}{}_{1}{}^{\alpha} (1 + k_{rr}{}_{1}{}^{\alpha})^{18} + PC_{F}{}_{2}{}^{\alpha} (1 + k_{rr}{}_{2}{}^{\alpha})^{17} + \ldots + PC_{F}{}_{23}{}^{\alpha} (1 + k_{rr}{}_{23}{}^{\alpha})^{1}$$

For $\alpha = 0$:

$$TV_{1}{}^{(\alpha=0)} = PC_{F}{}_{1}{}^{(\alpha=0)} (1 + k_{rr}{}_{1}{}^{(\alpha=0)})^{23} + PC_{F}{}_{2}{}^{(\alpha=0)} (1 + k_{rr}{}_{2}{}^{(\alpha=0)})^{22} + \ldots + PC_{F}{}_{23}{}^{(\alpha=0)} (1 + k_{rr}{}_{23}{}^{(\alpha=0)})^{1}$$

$$= 1,042.9x(1 + 0.117)^{18} + 1,268.2x(1 + 0.117)^{17} + \ldots + 37.4x(1 + 0.117)^{1}$$

$$= 7,642.82 + 8,319.4 + \ldots + 41.78$$

$$= 32,258.69$$

The values of the $TV_{\alpha}$ are shown in Table 10, for $\alpha$ values varying from 0 to 1.
period. Therefore,\( PV^{\alpha} \) can be calculated as shown below:

\[
PV^{\alpha} = [PV^{l(\alpha)}, PV^{r(\alpha)}] = [NCF_{0}^{l(\alpha)}, NCF_{0}^{r(\alpha)}] + ... + \left[ \frac{NCF_{l}^{l(\alpha)}}{(1 + k_{wacc}^{(l(\alpha))})^t}, \frac{NCF_{l}^{r(\alpha)}}{(1 + k_{wacc}^{(r(\alpha))})^t} \right] + ... \]

\[
+ \left[ \frac{NCF_{23}^{l(\alpha)}}{(1 + k_{wacc}^{(l(\alpha))})^{23}}, \frac{NCF_{23}^{r(\alpha)}}{(1 + k_{wacc}^{(r(\alpha))})^{23}} \right]
\]

where:

\[
PV^{l(\alpha)} = NCF_{0}^{l(\alpha)} + \frac{NCF_{l}^{l(\alpha)}}{(1 + k_{wacc}^{l(\alpha)})^t} + \frac{NCF_{23}^{l(\alpha)}}{(1 + k_{wacc}^{l(\alpha)})^{23}} + ... + \frac{NCF_{0}^{r(\alpha)}}{(1 + k_{wacc}^{r(\alpha)})^t} + \frac{NCF_{l}^{r(\alpha)}}{(1 + k_{wacc}^{r(\alpha)})^t} + \frac{NCF_{23}^{r(\alpha)}}{(1 + k_{wacc}^{r(\alpha)})^{23}} + ...
\]

For \( \alpha = 0 \):

\[
PV^{l(\alpha=0)} = NCF_{0}^{l(\alpha=0)} + \frac{NCF_{l}^{l(\alpha=0)}}{(1 + k_{wacc}^{(l(\alpha=0))})^t} + \frac{NCF_{23}^{l(\alpha=0)}}{(1 + k_{wacc}^{l(\alpha=0)})^{23}} + ... = -33.5 + \frac{-472.4}{(1 + 0.09)^t} + \frac{-1.054.3}{(1 + 0.09)^t} + ... + \frac{-495.0}{(1 + 0.09)^t} = -33.5 - 433.40 - 887.39 - ... - 62.20 = -2.582.35
\]

\[
PV^{r(\alpha=0)} = NCF_{0}^{r(\alpha=0)} + \frac{NCF_{l}^{r(\alpha=0)}}{(1 + k_{wacc}^{r(\alpha=0)})^t} + \frac{NCF_{23}^{r(\alpha=0)}}{(1 + k_{wacc}^{r(\alpha=0)})^{23}} + ... = -27.4 + \frac{-386.5}{(1 + 0.11)^t} + \frac{-386.8}{(1 + 0.11)^t} + ... + \frac{-405.0}{(1 + 0.11)^t} = -27.4 - 348.20 - 700.27 - ... - 36.73 = -2.007.35
\]

The \( PV^{\alpha} \) values, for different \( \alpha \) values varying from 0 to 1, are also shown in Table 10.

### Table 10
Fuzzy TV and PV values of the investment project (USS MM).

<table>
<thead>
<tr>
<th>( \alpha )</th>
<th>( PV^{\alpha} )</th>
<th>( TV^{\alpha} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>33,258.69</td>
<td>58,117.04</td>
</tr>
<tr>
<td>0.1</td>
<td>42,239.63</td>
<td>56,974.05</td>
</tr>
<tr>
<td>0.2</td>
<td>35,241.60</td>
<td>55,066.81</td>
</tr>
<tr>
<td>0.3</td>
<td>36,269.66</td>
<td>53,594.57</td>
</tr>
<tr>
<td>0.4</td>
<td>37,323.23</td>
<td>52,156.58</td>
</tr>
<tr>
<td>0.5</td>
<td>38,402.69</td>
<td>50,752.13</td>
</tr>
<tr>
<td>0.6</td>
<td>39,508.60</td>
<td>49,380.51</td>
</tr>
<tr>
<td>0.7</td>
<td>40,641.54</td>
<td>48,041.02</td>
</tr>
<tr>
<td>0.8</td>
<td>41,802.11</td>
<td>46,732.97</td>
</tr>
<tr>
<td>0.9</td>
<td>42,990.92</td>
<td>45,455.71</td>
</tr>
<tr>
<td>1.0</td>
<td>44,208.57</td>
<td>44,208.57</td>
</tr>
</tbody>
</table>

### Table 11
Fuzzy MNPV values (USS MM).

<table>
<thead>
<tr>
<th>( \alpha )</th>
<th>( MNPV^{\alpha} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>(1,044.75)</td>
</tr>
<tr>
<td>0.1</td>
<td>(927.21)</td>
</tr>
<tr>
<td>0.2</td>
<td>(805.07)</td>
</tr>
<tr>
<td>0.3</td>
<td>(678.06)</td>
</tr>
<tr>
<td>0.4</td>
<td>(545.92)</td>
</tr>
<tr>
<td>0.5</td>
<td>(408.36)</td>
</tr>
<tr>
<td>0.6</td>
<td>(265.08)</td>
</tr>
<tr>
<td>0.7</td>
<td>(115.75)</td>
</tr>
<tr>
<td>0.8</td>
<td>39.94</td>
</tr>
<tr>
<td>0.9</td>
<td>202.37</td>
</tr>
<tr>
<td>1.0</td>
<td>371.88</td>
</tr>
</tbody>
</table>

The fuzzy MIRR values are shown in Table 11 for different \( \alpha \) values varying from 0 to 1.

### 4.3.2.5. Calculation of fuzzy modified indices

Based on the fuzzy terminal values (TV\( ^{\alpha} \)) and fuzzy present values (PV\( ^{\alpha} \)), the fuzzy MNPV, MIRR, MPI, and MTPB indices are obtained as follows:

(a) Fuzzy Modified Net Present Value (MNPV\( ^{\alpha} \))

The fuzzy MNPV\( ^{\alpha} \) for \( \alpha = 0 \) is equal to:

\[
MNPV^{l(\alpha=0)} = \frac{TV^{l(\alpha=0)}}{(1 + k_{rad}^{l(\alpha=0)})^t} + PV^{l(\alpha=0)} = 33,258.69 + \frac{-472.4}{(1 + 0.09)^t} = 2582.35 = 1044.752
\]

\[
MNPV^{r(\alpha=0)} = \frac{TV^{r(\alpha=0)}}{(1 + k_{rad}^{r(\alpha=0)})^t} + PV^{r(\alpha=0)} = 58,117.04 + \frac{-386.5}{(1 + 0.11)^t} = 2007.35 = 2553.87
\]

The MNPV\( ^{\alpha} \) values are shown in Table 11 for different \( \alpha \) values varying from 0 to 1.

A graphic representation of the MNPV\( ^{\alpha} \) is shown in Fig. 6. The most promising present net value is $371.88. The highest and lowest values of the fuzzy MNPV are $2553.87 and $1044.75, respectively. According to the failure index of Chiu and Park (1994), the area with negative MNPV indicates that there is a possibility of approximately 21.4% for the project not being justified as acceptable.

(b) Fuzzy Modified Internal Rate of Return (MIRR\( ^{\alpha} \))

The fuzzy MIRR\( ^{\alpha} \) values are obtained based on Eqs. (28)–(31). Therefore, the MIRR\( ^{\alpha} \) for \( \alpha = 0 \) is equal to:

\[
MIRR^{l(\alpha=0)} = \left( \frac{TV^{l(\alpha=0)}}{PV^{l(\alpha=0)}} \right)^{1/t} - 1 = \left( \frac{33,258.69}{2,582.35} \right)^{1/23} - 1 = 0.1180\alpha11.8%
\]
Table 12
Fuzzy MIRR values (%).

<table>
<thead>
<tr>
<th>α</th>
<th>MIRR&lt;sup&gt;(α)&lt;/sup&gt;</th>
<th>MIRR&lt;sup&gt;(0)&lt;/sup&gt;</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>11.8</td>
<td>15.8</td>
</tr>
<tr>
<td>0.1</td>
<td>12.0</td>
<td>15.6</td>
</tr>
<tr>
<td>0.2</td>
<td>12.1</td>
<td>15.4</td>
</tr>
<tr>
<td>0.3</td>
<td>12.3</td>
<td>15.1</td>
</tr>
<tr>
<td>0.4</td>
<td>12.5</td>
<td>14.9</td>
</tr>
<tr>
<td>0.5</td>
<td>12.7</td>
<td>14.7</td>
</tr>
<tr>
<td>0.6</td>
<td>12.9</td>
<td>14.5</td>
</tr>
<tr>
<td>0.7</td>
<td>13.1</td>
<td>14.3</td>
</tr>
<tr>
<td>0.8</td>
<td>13.3</td>
<td>14.1</td>
</tr>
<tr>
<td>0.9</td>
<td>13.5</td>
<td>13.9</td>
</tr>
<tr>
<td>1.0</td>
<td>13.7</td>
<td>13.7</td>
</tr>
</tbody>
</table>

Table 13
Fuzzy MPI values.

<table>
<thead>
<tr>
<th>α</th>
<th>MPI&lt;sup&gt;α&lt;/sup&gt;</th>
<th>MPI&lt;sup&gt;(0)&lt;/sup&gt;</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.60</td>
<td>2.27</td>
</tr>
<tr>
<td>0.1</td>
<td>0.64</td>
<td>2.12</td>
</tr>
<tr>
<td>0.2</td>
<td>0.68</td>
<td>1.99</td>
</tr>
<tr>
<td>0.3</td>
<td>0.73</td>
<td>1.86</td>
</tr>
<tr>
<td>0.4</td>
<td>0.78</td>
<td>1.74</td>
</tr>
<tr>
<td>0.5</td>
<td>0.83</td>
<td>1.62</td>
</tr>
<tr>
<td>0.6</td>
<td>0.89</td>
<td>1.52</td>
</tr>
<tr>
<td>0.7</td>
<td>0.95</td>
<td>1.42</td>
</tr>
<tr>
<td>0.8</td>
<td>1.02</td>
<td>1.33</td>
</tr>
<tr>
<td>0.9</td>
<td>1.09</td>
<td>1.24</td>
</tr>
<tr>
<td>1.0</td>
<td>1.16</td>
<td>1.16</td>
</tr>
</tbody>
</table>

![Fig. 7. Fuzzy MIRR.](image)

The MIRR<sup>α</sup> values are shown in Table 12, for different α values varying from 0 to 1.

A graphic representation of the fuzzy MIRR is shown in Fig. 7. The most promising modified internal rate of return is 13.7% per year. The highest and lowest MIRR values are 15.8% per year and 11.8% per year, respectively. According to the failure index of Chiu and Park (1994), the area where the MIRR is lower than the k<sub>naive</sub> (13.0% per year) indicates that there is a possibility of approximately 19.0% that the project will not be justified as acceptable.

(c) Fuzzy Modified Profitability Index (MPI<sup>α</sup>)

The fuzzy MPI values are obtained based on Eqs. (32)–(35). Therefore, the MPI<sup>α</sup> for α = 0 is equal to:

\[
\text{MPI}^{\alpha}(\alpha=0) = \left[ -\frac{TV^{\alpha}(\alpha=0)}{PV^{\alpha}(\alpha=0)} \right]^{1/n} - 1 = \left[ \frac{58.117.04}{2.007.35} \right]^{1/23} - 1 = 0.158\text{ou}15.8\% 
\]

The MPI<sup>α</sup> values are shown in Table 13, for different α values varying from 0 to 1.

A graphic representation of the MPI<sup>α</sup> is shown in Fig. 8. The most promising modified profitability index is 1.16. The highest and lowest values of the MPI are 2.27 and 0.60, respectively. Again, according to the failure index of Chiu and Park (1994), the area with an MPI lower than 1 (one) indicates that there is a possibility of approximately 17.1% that the project will not be justified as acceptable.

(d) Fuzzy Modified Total Payback Period (MTPB<sup>α</sup>)

The fuzzy MTPB values are obtained based on Eqs. (36)–(39). The MTPB<sup>α</sup> for α = 0 is equal to:

\[
\text{MTPB}^{\alpha}(\alpha=0) = -PV^{\alpha}(\alpha=0) \times \left[ \frac{TV^{\alpha}(\alpha=0)}{(1 + k^{\alpha}(\alpha=0))} \right] \times n \\
= 2.007.35 \times \left[ \frac{58.117.04}{(1 + 0.117)^{23}} \right] = 10.12
\]

\[
\text{MTPB}^{\alpha}(\alpha=0) = -PV^{\alpha}(\alpha=0) \times \left[ \frac{TV^{\alpha}(\alpha=0)}{(1 + k^{\alpha}(\alpha=0))} \right] \times n \\
= 2.582.35 \times \left[ \frac{33.258.69}{(1 + 0.143)^{23}} \right] = 38.63
\]

The MTPB<sup>α</sup> values are shown in Table 14, for different values α varying from 0 to 1.

A graphic representation of the MTPB<sup>α</sup> is shown in Fig. 9. The most promising modified total payback period is 19.78 years. The highest and lowest MTPB values are 38.63 and 10.12 years, respectively. Based on the Chiu and Park (1994) failure index, the area with the highlighted MTPB indicates that there is a possibility...
of approximately 45.5% that the project will not be justified as acceptable.

4.4. Discussion of the results

Table 15 summarizes the results obtained by the application of the traditional methods (NPV, PI, and TPB), and the modified methods (MNPV, MIRR, MPI, and MTPB), deterministic and fuzzy, in the project for developing oil fields. The comparative analysis of the indices summarized in this table leads to the following conclusions:

• All indices—deterministic and fuzzy—lead to the acceptance of the project when considered as a high-risk project. However, through the fuzzy methods, it was possible to identify the possibility of failure of the project.
• The modified methods (MNPV, MIRR, MPI, and MTPB), differently from the traditional methods (NPV, IRR, PI, and TPB), allowed the elimination of the bias of the risk-adjusted discount rate, which overestimates (underestimates) the value of the traditional indices above (below) the result of the project.
• The use of triangular fuzzy numbers associated to the uncertainties of the project variables provides the possibility that the project will not be justified as an acceptable project (approximately 4%, for the fuzzy NPV, 21.4% for the fuzzy MNPV, 19.0% for the fuzzy MIRR, 17.1% for the fuzzy MPI, and 45.5% for the fuzzy MTPB). This was not possible to determine based on the deterministic indices, which simply indicate the acceptance of the project.
• It can be noted from Figs. 8 and 9 that there is a significant difference in the forms of the exact and approximate triangular fuzzy numbers, corresponding to the indices of the fuzzy MPI and fuzzy MTPB, respectively. This deviation is due to the high number of arithmetic operations undertaken in obtaining these indices. The use of the exact forms of the fuzzy numbers would result in failure indices of 18.0 and 40.3%, instead of the previously found values of 17.1% and 45.5%, respectively. Despite the fact that these new values do not change the result of the analysis, it is suggested that in future analyses the suitability of the use of the approximated forms of the triangular fuzzy numbers be verified on a case-by-case basis.
• A significant increase in the domains of the triangular fuzzy numbers can be observed in Figs. 6–9, corresponding to the fuzzy modified indicators. This is also due to the high number of arithmetic operations performed to obtain these indices. However, despite this increase, the conclusions provided by the proposed methods are not affected.

5. Conclusions

The main objective of this paper was to present a new unified fuzzy approach for the evaluation of investment projects under uncertainties, based on the modified deterministic methods of MIRR, MNPV, MPI and MTPB discussed by Sampaio Filho (2014). This unified approach explicitly predicts the use of the costs of opportunity associated with the intermediary cash flows of an investment project and eliminates the major problems of the traditional fuzzy and deterministic methods: the possibility of multiple rates of return of the IRR method; the implicit assumptions of the reinvestment rates of the IRR and NPV methods; and the anomalous behavior of the NPV method.

A modified fuzzy method was proposed and implemented for the modified net present value (fuzzy MNPV), the modified internal rate of return (fuzzy MIRR), the modified profitability index (fuzzy MPI), and the modified total payback period (fuzzy MTPB) for the analysis of investments in fuzzy conditions. Triangular fuzzy numbers were used to represent the uncertainties of some of the project variables: the cash flows, the financing rates, the reinvestment rates, and risk-adjusted discount rate.

Deterministic functions (MNPV, MPI, and MTPB) and fuzzy functions (NPVFuzzy, MNPVFuzzy, MIRRFuzzy, MPIFuzzy, and MTPBFuzzy) were developed in VBA of MS-Excel and used as a tool to support the calculations of the numeric examples, providing good performance and ease of use, which are essential characteristics for professional use.8

The new deterministic and fuzzy methods and their respective financial functions were tested in a case study, performed with data from a real application in the oil area, confirming the good applicability of the proposed models. In general terms, it can be concluded that:

• The application of costs of opportunities and fuzzy criteria for the valorization of uncertain variables produced results that were more compatible with market conditions.
• The proposed approach is simple and makes use of fewer hypotheses than the traditional risk treatment models do in terms of the behavior of the related variables. This is very useful to decision-makers.
• The replacement of the traditional methods (NPV, IRR, PI, and TPB) by the modified ones (MNPV, MIRR, MPI, and MTPB) helped eliminate biases of the risk-adjusted discount methodology, which overestimates (underestimates) the value of the traditional indices above (below) the result of the project.

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7 For a more detailed analysis, see Kaufmann and Gupta (1988).
8 The VBA syntaxes and codes of the functions (deterministic and fuzzy) are available in Sampaio Filho (2014).
All methods (MNPV, MIRR, MPI and MTPB) lead to consistent indications of acceptance or rejection of projects with unconventional cash flows.

The proposed fuzzy methods provide information about the possibility of failure of the project (which is not possible with the deterministic indices, which indicate the unconditional acceptance of the project).

The use of fuzzy sets concepts, together with the modified methods, proved to be adequate to solve capital budget problems for which the variables—net cash flows and discount rates—are defined in a non-deterministic way. This new approach assumes few hypotheses regarding the behavior of the related variables and the decision-maker has to put less effort in the adjustment of parameters. By offering different variation intervals of the indices for each of the different trust levels—instead of a single value—, this approach is more suited to the human reasoning inherent to decision makers.

Some authors, such as Buckley (1987a, 1987b) and Kahraman et al. (2002), argue that it is not possible to define a fuzzy IRR, since the calculation of the IRR (and the MIRR) requires that the NPV of a project be equal to zero. According to these authors, zero is a crisp number, whereas the NPV is a fuzzy number, which makes equality impossible. In this paper, an interpretation for the fuzzy MIRR was used, based on the representation by α-cuts, to work around this conflict.

Another controversial point, also related to the fuzzy IRR, is the possibility of the existence of multiple fuzzy internal return rates (González, Flores, Flores, & Mendoza, 2001). This conflict was avoided by using the fuzzy MIRR method instead of the traditional fuzzy IRR method.

### Table 15
Comparative analysis of the investment project indices.

<table>
<thead>
<tr>
<th>Traditional methods</th>
<th>Modified methods</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>NPV</strong> = $532.05</td>
<td><strong>MNPV</strong> = $371.88</td>
</tr>
<tr>
<td>IRR = 18.4% and −20.6% per annum (The existence of two discount rates makes inappropriate to use the IRR method).</td>
<td>MIRR = 13.7% per annum</td>
</tr>
<tr>
<td><strong>MPI</strong> = 1.16</td>
<td>MTPB = 19.8 years</td>
</tr>
</tbody>
</table>

**Fuzzy values**

<table>
<thead>
<tr>
<th>Traditional fuzzy values</th>
<th>Modified fuzzy values</th>
</tr>
</thead>
<tbody>
<tr>
<td>Worst value: −$1,044.75</td>
<td>Worst value: $213.75</td>
</tr>
<tr>
<td>Most promising value: $371.88</td>
<td>Most promising value: $532.05</td>
</tr>
<tr>
<td>Best value: $2553.87</td>
<td>Best value: $1331.51</td>
</tr>
<tr>
<td>Failure index: 21.4%</td>
<td>Failure index: 4%</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Traditional fuzzy values</th>
<th>Modified fuzzy values</th>
</tr>
</thead>
<tbody>
<tr>
<td>Worst value: 0.6</td>
<td>Worst value: 38.6</td>
</tr>
<tr>
<td>Most promising value: 1.16</td>
<td>Most promising value: 19.8</td>
</tr>
<tr>
<td>Best value: 2.27</td>
<td>Best value: 10.1</td>
</tr>
<tr>
<td>Failure index: 21.3%</td>
<td>Failure index: 45.5%</td>
</tr>
</tbody>
</table>
Some aspects have not been taken into account in this work and may be object of future developments. For example:

- The procedure used in this work treats positive and negative cash flows in an independent way. The possible occurrence of cash flows that are neither completely positive nor completely negative, as identified by Tercero et al. (2003), may be analyzed as a special case.

- In the case of correlated cash flows in a project, the variance is smaller than when the variables are independent from each other. A future work may consider that correlation between cash flows or between reinvestment rates, financing and risk adjusted discount.

- Fuzzy numbers ordering is one of the most controversial subjects in fuzzy literature. Chiu and Park (1994) and Tercero et al. (2003) refer to a series of papers that describe different procedures for ordering triangular fuzzy numbers and analyze which properties should be satisfied. The ordering of fuzzy indicators for mutually exclusive projects would be an interesting development.

- An increase in the domain of fuzzy numbers is a result of the quantity and nature of operations used in the computation of the fuzzy modified indicators. Methods for reducing the resulting variance (Carlsson, Fullér, & MEZEL 2012) may be used to improve the estimation of those indicators.

Finally, it is important to note that capital budget is, without doubt, one of the most important aspects in a company. The selection of investment projects that increase the wealth of the shareholders is not only one of the various activities of an organization, but one of its main goals. As a result, the process of identifying and selecting these projects is extremely important and cannot be a task aimed at only one method or criterion. The establishment of an ideal method such as traditional deterministic NPV is too simple and ignores the multidimensional nature of profitability. Each of the capital budget methods proposed in this paper meets a requirement. Each one provides an index and each has its limitations. None of these methods alone can completely discriminate between the alternatives of projects that differ in scale, lifespan, or cash flow patterns. Thus, a combination of methods that uses the strong points of each one, works around their conflicts and individual disadvantages, and also provides the possibility of incorporating options and preferences in the process of selecting investment alternatives, is a more adequate selection criterion than one based on the concept of individual method.

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References


